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# THE CATEGORY OF FINITELY PRESENTED SMOOTH MOD $p$ REPRESENTATIONS OF $GL_2(F)$

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**ABSTRACT.** Let  $F$  be a finite extension of  $\mathbb{Q}_p$ . We prove that the category of finitely presented smooth  $Z$ -finite representations of  $GL_2(F)$  over a finite extension of  $\mathbb{F}_p$  is an abelian subcategory of the category of all smooth representations. The proof uses amalgamated products of completed group rings.

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## 1 INTRODUCTION

Let  $\mathbb{F}$  be a finite field of characteristic  $p$ . If  $G$  is a locally profinite topological group, let  $\mathcal{C}_{\mathbb{F}}(G)$  be the category of smooth representations of  $G$  over  $\mathbb{F}$ . Throughout this paper, if  $K$  is an open subgroup of such a group  $G$  then  $\mathrm{ind}_K^G$  denotes induction with compact support modulo  $K$ .

**DEFINITION 1.1.** Let  $V$  be a smooth  $\mathbb{F}$ -representation of a locally profinite group  $G$ . Then  $V$  is:

1. **FINITELY GENERATED** if for some compact open subgroup  $K$  of  $G$  there is a surjection of  $\mathbb{F}[G]$ -modules

$$\mathrm{ind}_K^G W \twoheadrightarrow V$$

for a smooth finite-dimensional  $\mathbb{F}$ -representation  $W$  of  $K$ ;

2. FINITELY PRESENTED if for some compact open subgroups  $K_1, K_2$  of  $G$  there is an exact sequence

$$\mathrm{ind}_{K_1}^G W_1 \rightarrow \mathrm{ind}_{K_2}^G W_2 \rightarrow V \rightarrow 0$$

for  $W_1$  and  $W_2$  smooth finite-dimensional  $\mathbb{F}$ -representations of  $K_1$  and  $K_2$  respectively.

Let  $F$  be a finite extension of  $\mathbb{Q}_p$ . The purpose of this article is to prove:

**THEOREM 1.2.** *The category of finitely presented smooth  $\mathbb{F}$ -representations of  $SL_2(F)$  is an abelian subcategory of  $\mathcal{C}_{\mathbb{F}}(SL_2(F))$ .*

*The same holds for the category of finitely presented smooth  $Z$ -finite representations of  $GL_2(F)$ .*

This is Theorem 5.1 and Corollary 5.2 below. In fact, we prove the same result with  $F$  replaced by any finite dimensional division algebra over  $\mathbb{Q}_p$ .

The theorem is equivalent to the statement that the kernel<sup>1</sup> of any map between finitely presented smooth representations is itself finitely presented. If  $\mathcal{C}_{\mathbb{F}}(SL_2(F))$  were the category of modules over a ring  $R$ , this would be the statement that  $R$  is a coherent ring. Indeed, we will prove the theorem by considering smooth  $\mathbb{F}$ -representations as modules over the amalgamated product

$$\mathbb{F}[[K]] *_{\mathbb{F}[[I]]} \mathbb{F}[[K']],$$

where  $K = SL_2(\mathcal{O}_F)$ ,  $K' = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} K \begin{pmatrix} 1 & 0 \\ 0 & \pi^{-1} \end{pmatrix}$  for  $\pi$  a uniformising element of  $D$ , and  $I = K \cap K'$ . Then a result of Åberg [Å82] shows that, under certain conditions, an amalgamated product of coherent rings over a noetherian ring is itself coherent. *Throughout, unless otherwise stated, by ‘module’, ‘noetherian’ or ‘coherent’ we mean ‘left module’, ‘left noetherian’ or ‘left coherent’.*

Finitely presented representations of  $GL_2(F)$  were previously studied by Hu [Hu12], Vigneras [Vig11], and Schraen [Sch15].<sup>2</sup> In particular, [Vig11] Theorem 6 shows that a smooth *admissible* finitely presented representation of  $GL_2(F)$  has finite length, and that all of its subquotients are also admissible and finitely presented. On the other hand, the main result of [Sch15] says that, if  $F$  is a quadratic extension of  $\mathbb{Q}_p$ , then an irreducible supersingular representation of  $GL_2(F)$  admitting a central character is never finitely presented.

We are motivated by the construction (see [CEG<sup>+</sup>16]) of a ‘patched module’  $M_{\infty}$  that has an action of  $G = GL_n(F)$  and, hopefully, interpolates the hypothetical  $p$ -adic Langlands correspondence. It is (only?) possible to directly obtain information about  $M_{\infty}$  by considering  $\mathrm{Hom}_{GL_n(F)}(\mathrm{ind}_K^G(W), M_{\infty}^{\vee})$  for

<sup>1</sup>and the cokernel, but this is automatic

<sup>2</sup>The definition of ‘finitely presented’ in these articles is slightly different to ours, and automatically entails  $Z$ -finiteness.

locally algebraic representations of  $K = GL_n(\mathcal{O}_F)$  on finitely generated  $\mathbb{Z}_p$ -modules  $W$ . This leads us to consider the category of finitely presented representations of  $G$ ; it also motivates us to prove a version of Theorem 1.2 with coefficients.

I do not know whether Theorem 1.2 holds when  $G = GL_n(F)$  (or any  $p$ -adic Lie group). The method of this paper does not apply, because  $G$  is not (up to centre) an amalgam of two compact open subgroups. I am not sure whether Theorem 1.2 holds when  $F$  has positive characteristic; the method of this paper fails because  $GL_2(\mathcal{O}_F)$  is not  $p$ -adic analytic and its completed group ring is not noetherian. I thank Billy Woods for a helpful discussion about this case.

I am grateful to Matthew Emerton for asking me the question that this paper answers, and for several helpful and motivational conversations. I also thank Julien Hauseux and Stefano Morra for comments and corrections. I am indebted to the anonymous referee for suggesting that I relate the amalgamated product of rings considered here to the ring  $\Lambda(G)$  considered in [Koh17], which greatly clarified and simplified the arguments of this paper.

## 2 FINITELY PRESENTED REPRESENTATIONS

For the rest of this article, let  $\mathbb{F}$  be a finite field of characteristic  $p$ . Let  $A$  be a complete local noetherian  $W(\mathbb{F})$ -algebra with maximal ideal  $\mathfrak{m}$  and residue field  $\mathbb{F}$ . Let  $G$  be a locally profinite group. Recall ([Eme10] definition 2.2.5) that a *smooth*  $A$ -representation of  $G$  is a representation of  $G$  on a torsion  $A$ -module  $V$  such that every  $v \in V$  is fixed by a compact open subgroup of  $G$ .

DEFINITION 2.1. If  $K$  is a profinite group, then a *finite rank*  $A$ -representation of  $K$  is a representation of  $K$  on a finitely generated  $A$ -module  $M$  such that, for every  $n \geq 0$ ,  $M/\mathfrak{m}^n M$  is a smooth representation of  $K$ .

Strictly speaking, we should call these finite rank continuous  $A$ -representations of  $K$ .

DEFINITION 2.2. A representation of  $G$  on an  $A$ -module  $V$  is  *$K$ -finite* if for some (equivalently, any) compact open subgroup  $K \subset G$ , and for every  $v \in V$ , the  $A[K]$ -module generated by  $v$  is a finite rank  $A$ -representation of  $K$ .

We let  $\mathcal{C}_A^{K\text{-fin}}(G)$  be the category of all  $K$ -finite  $A$ -representations of  $G$ , with morphisms being morphisms of  $A[G]$ -modules. Note that a representation of  $G$  on a *torsion*  $A$ -module  $V$  is smooth if and only if it is  $K$ -finite.

In the introduction (Definition 1.1) we gave the definitions of ‘finitely generated’ and ‘finitely presented’ smooth  $\mathbb{F}$ -representations of  $G$ . We now extend those to  $K$ -finite  $A$ -representations. First, note that if  $M$  is a finite rank  $A$ -representation of a compact open subgroup  $K \subset G$ , then  $\text{ind}_K^G M$  is certainly  $K$ -finite.

DEFINITION 2.3. Let  $V$  be a  $K$ -finite  $A$ -representation of  $G$ . Then  $V$  is:

1. *finitely generated* if there is a compact open subgroup  $K \subset G$ , a finite rank  $A$ -representation  $W$  of  $K$ , and a surjection of  $A[G]$ -modules

$$\mathrm{ind}_K^G(W) \twoheadrightarrow V;$$

2. *finitely presented* if for some compact open subgroups  $K_1, K_2$  of  $G$  there is an exact sequence of  $A[G]$ -modules

$$\mathrm{ind}_{K_1}^G W_1 \rightarrow \mathrm{ind}_{K_2}^G W_2 \rightarrow V \rightarrow 0$$

for  $W_1$  and  $W_2$  finite rank  $A$ -representations of  $K_1$  and  $K_2$ .

We start by establish some straightforward properties of finitely presented  $K$ -finite representations. Many of the proofs follow those of the properties of finitely presented modules over a ring given in [Sta17, Tag 0519].

LEMMA 2.4. *A  $K$ -finite  $A$ -representation  $V$  of  $G$  is finitely generated if and only if it is finitely generated as an  $A[G]$ -module.*

*Proof.* For any  $W$  and  $K$ ,  $\mathrm{ind}_K^G W$  is generated (as an  $A[G]$ -module) by the finitely generated  $A$ -submodule of functions supported on  $K$ . The ‘only if’ direction follows.

For the ‘if’ direction, let  $V$  be a  $K$ -finite representation generated by  $v_1, \dots, v_n$  as an  $A[G]$ -module. Choose a compact open subgroup  $K$  and let  $W$  be the finite rank  $A$ -representation of  $K$  generated by  $v_1, \dots, v_n$ . Then  $V$  is a quotient of  $\mathrm{ind}_K^G W$ .  $\square$

REMARK 2.5. It is not true that a finitely presented  $K$ -finite  $A$ -representation of  $G$  will be finitely presented as an  $A[G]$ -module; this is already false for the  $\mathbb{F}$ -representation  $\mathrm{ind}_K^G \mathbb{F}$ , as long as  $K$  is not finitely generated. This is the main technical problem that we have to overcome in the next section.

LEMMA 2.6. *Suppose that  $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$  is a short exact sequence of  $K$ -finite  $A$ -representations of  $G$ .*

*If  $V_1$  and  $V_3$  are finitely generated, so is  $V_2$ .*

*Proof.* This is immediate from Lemma 2.4 and the fact that an extension of finitely generated modules over  $A[G]$  is finitely generated.  $\square$

LEMMA 2.7. *Suppose that  $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$  is a short exact sequence of  $K$ -finite  $A$ -representations of  $G$ .*

1. *If  $V_2$  is finitely presented and  $V_1$  is finitely generated, then  $V_3$  is finitely presented.*
2. *If  $V_3$  is finitely presented and  $V_2$  is finitely generated, then  $V_1$  is finitely generated.*
3. *If  $V_1$  and  $V_3$  are finitely presented, so is  $V_2$ .*

*Proof.* We use  $K$  and  $L, M, N$  to denote a suitably chosen compact open subgroup of  $G$  and finite rank  $A$ -representations of  $K$ .

1. Choose a presentation  $\text{ind}_K^G N \xrightarrow{\alpha} \text{ind}_K^G M \rightarrow V_2 \rightarrow 0$  and choose  $v_1, \dots, v_r$  generating the image of  $V_1$  in  $V_2$  as an  $A[G]$ -module. For each  $i$ , let  $\tilde{v}_i$  be a lift of  $v_i$  to  $\text{ind}_K^G M$ , and let  $L$  be the finite rank  $A$ -representation of  $K$  generated by the  $\tilde{v}_i$ . Then we have a map  $\gamma : \text{ind}_K^G L \rightarrow \text{ind}_K^G M$ , and the kernel of the (surjective) composition  $\text{ind}_K^G M \rightarrow V_2 \rightarrow V_3$  is the sum of the image of  $\alpha$  and the image of  $\gamma$ , and so is finitely generated.
2. Choose a presentation  $\text{ind}_K^G N \rightarrow \text{ind}_K^G M \xrightarrow{\alpha} V_3 \rightarrow 0$ . We may replace  $M$  by its image in  $V_3$ , so that we have  $M \subset V_3$  and  $\text{ind}_K^G M \rightarrow V_3$  is the natural map. Let  $m_1, \dots, m_r$  generate  $M \subset V_3$  as an  $A$ -module, and for each  $i$  let  $\tilde{m}_i \in V_2$  be a lift of  $m_i$ . Let  $\tilde{M}$  be the  $A[K]$ -span of the  $\tilde{m}_i$  in  $V_2$ . Then there is a surjective map of  $K$  representations  $\tilde{M} \rightarrow M$ , and we let  $L$  be the kernel. There is also a map  $\beta : \text{ind}_K^G \tilde{M} \rightarrow V_2$  giving a commuting diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & & \text{ind}_K^G N & & \\
 & & & & \downarrow & & \\
 \text{ind}_K^G L & \longrightarrow & \text{ind}_K^G \tilde{M} & \longrightarrow & \text{ind}_K^G M & \longrightarrow & 0 \\
 & & \beta \downarrow & & \downarrow \alpha & & \\
 & & V_2 & \longrightarrow & V_3 & \longrightarrow & 0.
 \end{array}$$

Repeating the same argument, we may replace  $N$  by an  $A[K]$ -submodule of  $\text{ind}_K^G M$  and find a  $K$ -submodule  $\tilde{N} \subset \text{ind}_K^G \tilde{M}$ , together with a surjection  $\tilde{N} \rightarrow N$  of  $A[K]$ -modules, such that

$$\begin{array}{ccc}
 \text{ind}_K^G \tilde{N} & \longrightarrow & \text{ind}_K^G N \\
 \downarrow & & \downarrow \\
 \text{ind}_K^G \tilde{M} & \longrightarrow & \text{ind}_K^G M.
 \end{array}$$

commutes and has surjective horizontal maps. The kernel of  $\text{ind}_K^G \tilde{M} \rightarrow V_3$  is the image of  $\text{ind}_K^G(\tilde{N} \oplus L)$ . Write  $\gamma$  for the restriction of  $\beta$  to  $\text{ind}_K^G(\tilde{N} \oplus L)$ . We obtain a commutative diagram

$$\begin{array}{ccccccc}
 \text{ind}_K^G(\tilde{N} \oplus L) & \longrightarrow & \text{ind}_K^G \tilde{M} & \longrightarrow & V_3 & \longrightarrow & 0 \\
 \gamma \downarrow & & \beta \downarrow & & \parallel & & \\
 0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 \longrightarrow 0
 \end{array}$$

with exact rows, from which we see that  $\text{cok}(\gamma) \cong \text{cok}(\beta)$ . As  $V_2$  is finitely generated, so is  $\text{cok}(\beta)$  and hence also  $\text{cok}(\gamma)$ . Since  $\text{im}(\gamma)$  is also finitely generated, we see that  $V_1$  is finitely generated by Lemma 2.6.

3. Choose surjections  $\alpha : \text{ind}_K^G M \rightarrow V_1$  and  $\beta : \text{ind}_K^G N \rightarrow V_3$ . As before, we may assume that  $N \subset V_3$ . Let  $n_1, \dots, n_r$  generate  $N$  as an  $A$ -module, lift them to  $\tilde{n}_i \in V_2$ , and let  $\tilde{N}$  be the  $A[K]$ -module generated by the  $\tilde{n}_i$ . Let  $\gamma$  be the resulting map  $\text{ind}_K^G \tilde{N} \rightarrow V_2$ . If we let  $L = \ker(\tilde{N} \rightarrow N)$ , then  $\gamma$  restricts to a map  $\gamma' : \text{ind}_K^G L \rightarrow V_1$ . We obtain a commuting diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{ind}_K^G(M \oplus L) & \longrightarrow & \text{ind}_K^G(M \oplus \tilde{N}) & \longrightarrow & \text{ind}_K^G N \longrightarrow 0 \\
 & & \alpha + \gamma' \downarrow & & \alpha + \gamma \downarrow & & \downarrow \beta \\
 0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 \longrightarrow 0
 \end{array}$$

with exact rows and surjective vertical maps. By the snake lemma there is a short exact sequence

$$0 \rightarrow \ker(\alpha + \gamma') \rightarrow \ker(\alpha + \gamma) \rightarrow \ker(\beta) \rightarrow 0.$$

Since the outer two terms are finitely generated by (2), so is the inner term (by Lemma 2.6). Thus  $V_2$  is finitely presented, as required.  $\square$

LEMMA 2.8. *Suppose that  $G' \subset G$  is a finite index open subgroup. Then a  $K$ -finite  $A$ -representation  $V$  of  $G$  is finitely generated/presented if and only if its restriction to  $G'$  is.*

*Proof.* 1. If  $V$  is finitely generated as a representation of  $G'$  then it certainly is as a representation of  $G$ . Conversely, for any compact open subgroup  $K$  of  $G$  and any finite rank  $A$ -representation  $W$  of  $K$ , we have the Mackey formula

$$\text{res}_{G'}^G \text{ind}_K^G W \cong \bigoplus_{g \in G' \backslash G/K} \text{ind}_{gKg^{-1} \cap G'}^{G'} W^g.$$

So  $\text{ind}_K^G W$  is finitely generated — in fact finitely presented — as a representation of  $G'$ . It follows that any finitely generated representation of  $G$  is finitely generated as a representation of  $G'$ .

2. We showed in (1) that  $\text{ind}_K^G W$  is finitely presented as a representation of  $G'$  for any finite rank  $A$ -representation  $W$  of a compact open subgroup  $K$ . It follows from Lemma 2.7 (1) that any  $K$ -finite finitely presented representation of  $G$  is finitely presented as a representation of  $G'$ .

Conversely, suppose that  $V$  is finitely presented as a representation of  $G'$ . By the first part, it is finitely generated as a representation of  $G$ , so that there is a surjection  $\text{ind}_K^G W \rightarrow V$ . Since the first term is finitely generated as a representation of  $G'$  by part (1), by Lemma 2.7 (2) the kernel is finitely generated as a representation of  $G'$ , and hence also as a representation of  $G$ . Therefore  $V$  is finitely presented as a representation of  $G$  by Lemma 2.7 (1).  $\square$

### 2.1 $Z$ -FINITENESS

Suppose that  $G$  is a locally profinite group with centre  $Z$ . We say that *Hypothesis  $Z$  is satisfied* if, for some (equivalently, any) compact open subgroup  $K$  of  $G$ ,  $Z/K \cap Z$  is finitely generated. Recall from [Eme10] the definitions of  $Z$ -finite and locally  $Z$ -finite representations: a representation is  $Z$ -finite if the action of  $A[Z]$  on  $V$  factors through a quotient  $A[Z]/I$  that is a finitely generated  $A$ -module. It is locally  $Z$ -finite if the  $A[Z]$ -module spanned by any  $v \in V$  is a finitely generated  $A$ -module. By [Eme10] Lemma 2.3.3, a representation of  $G$ , finitely generated as an  $A[G]$ -module, is  $Z$ -finite if and only if it is locally  $Z$ -finite.

LEMMA 2.9. *Let  $V$  be a locally  $Z$ -finite,  $K$ -finite,  $A$ -representation of  $G$ .*

1. *The representation  $V$  is finitely generated if and only if there is a surjection*

$$\mathrm{ind}_{KZ}^G W \rightarrow V \rightarrow 0$$

*for some compact open subgroup  $K$  of  $G$  and finite rank  $A$ -representation  $W$  of  $KZ$ .*

2. *If the representation  $V$  is finitely presented then there is an exact sequence*

$$\mathrm{ind}_{K_1Z}^G W_1 \rightarrow \mathrm{ind}_{K_2Z}^G W_2 \rightarrow V \rightarrow 0$$

*for some compact open subgroup  $K$  of  $G$  and finite rank  $A$ -representations  $W_1$  and  $W_2$  of  $K_1Z$  and  $K_2Z$ . If Hypothesis  $Z$  is satisfied, the converse holds.*

*Proof.* 1. The backwards implication is clear. For the forwards implication, let  $W$  be the  $A[KZ]$ -span of a finite set of generators of  $V$ . It is finite-rank since  $V$  is  $K$ -finite and locally  $Z$ -finite. We therefore get a surjection  $\mathrm{ind}_{KZ}^G W \rightarrow V \rightarrow 0$  as required.

2. Suppose that  $V$  is finitely presented. Then there is a surjection  $\mathrm{ind}_{KZ}^G W_2 \rightarrow V \rightarrow 0$ , by the first part. The kernel is finitely generated by Lemma 2.7 (2), and  $\mathrm{ind}_{KZ}^G W_2$  is  $Z$ -finite. Applying the first part again, we get an exact sequence  $\mathrm{ind}_{KZ}^G W_1 \rightarrow \mathrm{ind}_{KZ}^G W_2 \rightarrow V \rightarrow 0$  as required.

For the other direction, it is enough to show that (under Hypothesis  $Z$ )  $\mathrm{ind}_{KZ}^G W_2$  is finitely presented for any representation  $W_2$  of  $KZ$  on a finitely generated  $A$ -module. If  $U$  is the kernel of the natural map  $\mathrm{ind}_K^{KZ} W_2 \rightarrow W_2$  then there is a short exact sequence

$$0 \rightarrow \mathrm{ind}_{KZ}^G U \rightarrow \mathrm{ind}_K^G W_2 \rightarrow \mathrm{ind}_{KZ}^G W_2 \rightarrow 0.$$

We have to show that  $U$  is finitely generated as a  $KZ$ -representation. This follows from Hypothesis  $Z$ , since this implies that  $A[KZ/K]$  is a noetherian ring.  $\square$



Now suppose that  $H$  is an open subgroup of  $G$  such that  $HZ$  has finite index in  $G$  and  $Z \cap H$  is compact.

PROPOSITION 2.10. *Let  $V$  be a locally  $Z$ -finite,  $K$ -finite,  $A$ -representation of  $G$ .*

1. *The representation  $V$  of  $G$  is finitely generated if and only if its restriction to  $H$  is finitely generated.*
2. *If the representation  $V$  of  $G$  is finitely presented then its restriction to  $H$  is finitely presented. If Hypothesis  $Z$  holds, then the converse is true.*

*Proof.* By Lemma 2.8 we may assume that  $G = HZ$ . Let  $V$  be a locally  $Z$ -finite  $K$ -finite representation of  $G$ .

1. If  $V$  is finitely generated as a representation of  $H$ , it certainly is as a representation of  $G$ . Conversely, suppose that  $V$  is finitely generated as a representation of  $G$ . If  $W \subset V$  is a finitely generated  $A$ -module that generates  $V$  as a representation of  $G$ , then the  $Z$ -span  $ZW$  is a finitely generated  $A$ -module that generates  $V$  as a representation of  $H$ . So  $V$  is a finitely generated representation of  $H$  as required.
2. Suppose that  $V$  is finitely presented as a representation of  $G$ . By Lemma 2.9 (2) and Lemma 2.7 (1), it suffices to show that  $\text{ind}_{KZ}^G W$  is a finitely presented representation of  $H$  for  $K \subset H$ . This follows from the identity of representations of  $H$

$$\text{ind}_{KZ}^{HZ} W = \text{ind}_{K(Z \cap H)}^H W$$

and the assumption that  $Z \cap H$  is compact.

Finally, suppose that  $V$  is finitely presented as a representation of  $H$  and that Hypothesis  $Z$  holds. Then  $V$  is finitely generated as a representation of  $G$ , so by Lemma 2.9 (1) there is a surjection  $\text{ind}_{KZ}^G W \rightarrow V$ . By (1) and Lemma 2.7 (2) the kernel of this map is a finitely generated representation of  $H$ . By (1) again, it is a finitely generated representation of  $G$ , and so by Lemma 2.9 (1) we have an exact sequence

$$\text{ind}_{KZ}^G U \rightarrow \text{ind}_{KZ}^G W \rightarrow V \rightarrow 0.$$

As  $G$  satisfies Hypothesis  $Z$ , by the converse direction of Lemma 2.9 (2),  $V$  is a finitely presented representation of  $G$ .  $\square$

### 3 COMPLETED GROUP RINGS

If  $K$  is a profinite group, let

$$A[[K]] = \varprojlim_{J \triangleleft K_{\text{open}}} A[K/J]$$

be the completed group ring, a compact topological  $A$ -algebra.

LEMMA 3.1. *Suppose that  $M$  is a finite rank  $A$ -representation of  $K$ . Then there is a unique  $A[[K]]$ -module structure on  $M$  extending the  $A[K]$ -module structure.*

*Proof.* For each  $n$ , the action of  $A[K]$  on  $M \otimes_A A/\mathfrak{m}_A^n$  factors through  $A[K/J_n]$  for some open subgroup  $J_n \subset K$  and so extends uniquely to an action of  $A[[K]]$ . Since  $M$  is finitely generated as an  $A$ -module,  $M = \varprojlim M \otimes_A A/\mathfrak{m}_A^n$  and the lemma follows.  $\square$

Kohlhaase [Koh17] has extended the notion of completed group ring beyond the compact case. Let  $G$  be a locally profinite group.

PROPOSITION 3.2 (Kohlhaase). *If  $K \subset G$  is a compact open subgroup, then there is a unique  $A$ -algebra structure on*

$$A\langle G \rangle = A[G] \otimes_{A[K]} A[[K]]$$

*such that the natural maps  $A[G] \rightarrow A\langle G \rangle$  and  $A[[K]] \rightarrow A\langle G \rangle$  are  $A$ -algebra homomorphisms. This  $A$ -algebra is independent of the choice of  $K$  up to canonical isomorphism.*

*Proof.* This is shown in section 1 of [Koh17] when  $A$  is a field — where what we call  $A\langle G \rangle$  is denoted  $\Lambda(G)$  — but the proof works verbatim for general rings  $A$ . We recall the construction for the reader's convenience. Firstly, if  $K'$  is an open subgroup of  $K$ , then the natural map of  $(A[G], A[[K']])$ -bimodules

$$\rho_{K,K'} : A[G] \otimes_{A[K']} A[[K']] \rightarrow A[G] \otimes_{A[K]} A[[K]]$$

is an isomorphism.<sup>3</sup> If  $K'' \subset K'$  then we have  $\rho_{K,K''} = \rho_{K',K''} \circ \rho_{K,K'}$ , and so we may construct the direct limit

$$A\langle G \rangle = \varinjlim_K (A[G] \otimes_{A[K]} A[[K]])$$

which is (canonically) isomorphic to any one of its terms. Now, if  $g \in G$  then there is an isomorphism of direct systems

$$\cdot g : A[G] \otimes_{A[K]} A[[K]] \rightarrow A[G] \otimes_{A[g^{-1}Kg]} A[[g^{-1}Kg]]$$

taking  $h \otimes \kappa$  to  $hg \otimes g^{-1}\kappa g$ . This defines a right action of  $G$  on  $A\langle G \rangle$  by left  $A[G]$ -module isomorphisms, which suffices to define the required ring structure on  $A\langle G \rangle$ . Precisely, if  $h \otimes \kappa \in A[G] \otimes_{A[K]} A[[K]]$  and  $h' \otimes \kappa' \in A[G] \otimes_{A[K']} A[[K']]$  are representatives of elements of  $A\langle G \rangle$ , we may assume that  $K \subset h^{-1}K'h$  and define

$$(h' \otimes \kappa')(h \otimes \kappa) = h'h \otimes h^{-1}\kappa'h\kappa \in A[G] \otimes_{A[h^{-1}K'h]} A[[h^{-1}K'h]]. \quad \square$$

<sup>3</sup>In [Koh17] this is stated for  $K'$  normal in  $K$ , but it is true for any  $K'$  and moreover this is necessary for the construction of the ring structure.

For later use, we record a flatness result:

LEMMA 3.3. *The  $A$ -algebra  $A\langle G \rangle$  is flat as a right  $A[[K]]$ -module for any compact open subgroup  $K$  of  $G$ .*

*Proof.* As in [Koh17],  $A\langle G \rangle = A[G] \otimes_{A[K]} A[[K]] \cong \bigoplus_{h \in G/K} A[[K]]$  as right  $A[[K]]$ -modules, so that  $A\langle G \rangle$  is even a free right  $A[[K]]$ -module.  $\square$

REMARK 3.4. In the same way we could put an  $A$ -algebra structure on  $A[[K]] \otimes_{A[K]} A[G]$  (for any compact open subgroup  $K$ ) and the  $A$ -module map  $A[[K]] \otimes_{A[K]} A[G] \rightarrow A\langle G \rangle$  defined by

$$\kappa \otimes h \mapsto h \otimes h^{-1}\kappa h \in A[G] \otimes_{A[h^{-1}Kh]} A[[h^{-1}Kh]]$$

is an isomorphism of  $A$ -algebras. Thus Lemma 3.3 holds with ‘right’ replaced by ‘left’.

LEMMA 3.5. *Suppose that  $V$  is a  $K$ -finite  $A$ -representation of  $G$ . Then there is a unique  $A\langle G \rangle$ -module structure on  $V$  extending the  $A[G]$ -module structure.*

*Proof.* Since  $V$  is  $K$ -finite, for any compact open subgroup  $K$  the action of  $A[K]$  extends uniquely to an action of  $A[[K]]$  by Lemma 3.1. By the unicity, we have that, for any  $h \in G$  and  $\kappa \in A[[K]]$ , the two actions of  $h^{-1}\kappa h$  defined on the one hand by the actions of  $G$  and  $A[[K]]$ , and on the other hand by the action of  $A[[h^{-1}Kh]]$ , agree. From the formula for multiplication in  $A\langle G \rangle$  given in Proposition 3.2, it follows that we can define an action of  $A\langle G \rangle$  on  $V$  by fixing  $K$  and setting

$$(h \otimes \kappa)(v) = h(\kappa(v))$$

for any  $h \in A[G]$  and  $\kappa \in A[[K]]$ , which is clearly the unique action extending those of  $A[G]$  and  $A[[K]]$ .  $\square$

LEMMA 3.6. *Suppose that  $V$  is a  $K$ -finite  $A$ -representation of  $G$ . Then  $V$  is finitely generated if and only if it is finitely generated as an  $A\langle G \rangle$ -module.*

*Proof.* Suppose that  $V$  is finitely generated. By Lemma 2.4,  $V$  is finitely generated as an  $A[G]$ -module, and hence as a  $A\langle G \rangle$ -module.

Conversely, let  $V$  be a  $K$ -finite  $A$ -representation of  $G$  that is finitely generated as a  $A\langle G \rangle$ -module. Then, if  $v_1, \dots, v_r$  generate  $V$  and if  $M$  is their  $A[K]$ -span, then  $M$  is also preserved by  $A[[K]]$  and so

$$V = A\langle G \rangle \cdot M = (A[G] \otimes_{A[K]} A[[K]])M = A[G]M.$$

Therefore  $V$  is finitely generated, as required.  $\square$

The key technical reason for us to introduce the ring  $A\langle G \rangle$  is that it is true that a finitely presented  $K$ -finite  $A$ -representation of  $G$  is a finitely presented  $A\langle G \rangle$ -module — see Remark 2.5. The starting point is the following result of Lazard (see [Eme10] Theorem 2.1.1).

**THEOREM 3.7.** *If  $G$  is a  $p$ -adic analytic group, then  $A[[K]]$  is noetherian for every compact open subgroup  $K$  of  $G$ .  $\square$*

**PROPOSITION 3.8.** *Suppose that  $G$  is a  $p$ -adic analytic group. Let  $V$  be a  $K$ -finite  $A$ -representation of  $G$ . Then  $V$  is finitely presented if and only if it is finitely presented as an  $A\langle G \rangle$ -module.*

*Proof.* The backwards implication follows from Lemma 3.6. Suppose that  $V$  is finitely presented as an  $A\langle G \rangle$ -module. Then by Lemma 3.6 there is a surjection

$$\alpha : \operatorname{ind}_K^G W \rightarrow V \rightarrow 0$$

for some finite rank  $A$ -representation  $W$  of a compact open subgroup  $K \subset G$ . The kernel of  $\alpha$  is a  $K$ -finite representation of  $G$  that is finitely generated as an  $A\langle G \rangle$ -module, by [Sta17, Tag 0519] (5).<sup>4</sup> Therefore it is finitely generated as an  $A$ -representation of  $G$ , by Lemma 3.6.

Suppose now that  $V$  is finitely presented. Then by Lemma 2.7 (2), there is a compact open subgroup  $K$ , a finite rank  $A$ -representation  $M$  of  $K$ , and a surjection  $\operatorname{ind}_K^G M \rightarrow V \rightarrow 0$  with finitely generated kernel.

By [Sta17, Tag 0519] (4) and Lemma 3.6, it is enough to show that  $\operatorname{ind}_K^G(M)$  is a finitely presented  $A\langle G \rangle$ -module for the  $A\langle G \rangle$ -module structure provided by Lemma 3.5. We may think of this instead as the tensor product

$$\operatorname{ind}_K^G(M) \cong A[G] \otimes_{A[K]} M$$

via the isomorphism sending an element  $f : G \rightarrow M$  of  $\operatorname{ind}_K^G(M)$  to  $\sum_{g \in G/K} g \otimes f(g^{-1})$ . By Lemma 3.1 the action of  $A[K]$  on  $M$  extends uniquely to one of  $A[[K]]$  and we have isomorphisms

$$A\langle G \rangle \otimes_{A[[K]]} M = A[G] \otimes_{A[K]} \otimes_{A[[K]]} M = A[G] \otimes_{A[K]} M$$

of  $A[G]$ -modules, and hence of  $A\langle G \rangle$ -modules (by Lemma 3.5).

Since  $A[[K]]$  is noetherian by Theorem 3.7, the finitely generated  $A[[K]]$ -module  $M$  is finitely presented; let  $A[[K]]^m \rightarrow A[[K]]^n \rightarrow M \rightarrow 0$  be a presentation. Applying  $A\langle G \rangle \otimes_{A[[K]]} -$ , we obtain an exact sequence

$$A\langle G \rangle^n \rightarrow A\langle G \rangle^m \rightarrow A\langle G \rangle \otimes_{A[[K]]} M = A[G] \otimes_{A[K]} M \rightarrow 0$$

so that  $\operatorname{ind}_K^G M = A[G] \otimes_{A[K]} M$  is a finitely presented  $A\langle G \rangle$ -module, as required.  $\square$

#### 4 AMALGAMATIONS AND COHERENCE

Let  $K_1, K_2$  and  $I$  be profinite groups equipped with inclusions  $f_i : I \hookrightarrow K_i$  of  $I$  as a common open subgroup of  $K_1$  and  $K_2$ . Then there are maps  $f_i : A[[I]] \rightarrow A[[K_i]]$  of topological augmented  $A$ -algebras.

<sup>4</sup>Strictly speaking, [Sta17, Tag 0519] is only stated for modules over commutative rings. However, it is still true, with an identical proof, in the non-commutative case.

Let  $H = K_1 *_I K_2$  be the amalgamation of  $K_1$  and  $K_2$  along  $I$ . By [Ser77], Théorème 1, the natural map  $I \rightarrow H$  is injective. The following proposition shows that  $H$  is naturally a locally profinite topological group:

PROPOSITION 4.1. *With the colimit topology,<sup>5</sup>  $H$  is a locally profinite group with a basis of open neighbourhoods of the identity being given by open neighbourhoods of  $I$ .*

*Proof.* Let  $H$  and  $H'$  respectively denote  $H$  with the colimit topology and the topology for which translates of open subgroups of  $I$  are a basis of open sets. Let  $i : H \rightarrow H'$  and  $j : H' \rightarrow H$  be the identity maps; we have to show that they are both continuous. But  $i$  is continuous by the universal property of  $H$ , and  $j$  is continuous because the map  $I \rightarrow H$  is continuous.  $\square$

We now consider the amalgamated product of rings,  $A[[K_1]] *_A[[I]] A[[K_2]]$ . Note first that  $A[K_1] *_A[[I]] A[K_2]$  is simply the group ring of  $H$  over  $A$ . This is because the functor  $G \mapsto A[G]$  from groups to  $A$ -algebras is a left-adjoint, and so commutes with the colimit  $*$ .

In general, we have  $A$ -algebra maps  $A[[K_1]] \rightarrow A\langle H \rangle$  and  $A[[K_2]] \rightarrow A\langle H \rangle$  which agree on  $A[[I]]$ , and so (by the universal property) an  $A$ -algebra map  $\alpha : A[[K_1]] *_A[[I]] A[[K_2]] \rightarrow A\langle H \rangle$ .

PROPOSITION 4.2. *The map*

$$\alpha : A[[K_1]] *_A[[I]] A[[K_2]] \rightarrow A\langle H \rangle$$

*is an isomorphism of  $A$ -algebras.*

*Proof.* Let  $R = A[[K_1]] *_A[[I]] A[[K_2]]$ . The composite map

$$A[H] = A[K_1] *_A[[I]] A[K_2] \rightarrow R \rightarrow A\langle H \rangle$$

is easily seen to be the natural map  $A[H] \rightarrow A\langle H \rangle$ . It follows that the image of  $R$  in  $A\langle H \rangle$  contains  $A[H]$  and  $A[[K_i]]$  and so in fact is all of  $A\langle H \rangle$ , whence  $\alpha$  is surjective.

Moreover, from the universal property of  $\otimes$ , for each  $i$  we have a map of  $(A[H], A[[K_i]])$ -bimodules

$$A[H] \otimes_{A[K_i]} A[[K_i]] \rightarrow R$$

and these define the *same* map  $\beta : A\langle H \rangle \rightarrow R$ . Since this is a map of right  $A[[K_1]]$ - and  $A[[K_2]]$ -modules, we see that  $\beta \circ \alpha$  is the identity — it is enough to check that it takes 1 to 1. Therefore  $\alpha$  is injective and so an isomorphism.  $\square$

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<sup>5</sup>The coarsest topology on  $H$  such that for every topological group  $G$  equipped with continuous maps  $K_i \rightarrow G$  agreeing on  $I$ , there is a continuous map  $H \rightarrow G$  extending these.

## 4.1 COHERENCE

Recall that a ring  $R$  is (left) coherent if any of the following equivalent definitions hold:

1. every finitely generated left ideal of  $R$  is finitely presented;
2. if  $f : M \rightarrow N$  is a map of finitely presented left  $R$ -modules, then  $\ker(f)$  is finitely presented;
3. the category of finitely presented left  $R$ -modules is an abelian subcategory of the category of left  $R$ -modules.

PROPOSITION 4.3. *If the rings  $A[[K_i]]$  are coherent and  $A[[I]]$  is noetherian, then  $A\langle H \rangle$  is coherent.*

*Proof.* This follows immediately from [Å82] Theorem 12; the hypotheses of that theorem are satisfied, by Lemma 3.3. For the convenience of the reader, we summarise the argument of [Å82] in the case of interest to us. It uses the characterisation — due to Chase [Cha60] — of left coherent rings as those for which arbitrary products of right flat modules are flat. Let  $R$ ,  $S$  and  $T$  be rings such that  $S$  and  $T$  are  $R$ -algebras, and  $Q = S *_R T$  is flat as a right  $R$ ,  $S$  or  $T$ -module; we will take  $R = A[[I]]$  and  $S = A[[K_1]]$ ,  $T = A[[K_2]]$ . Then there is a Mayer–Vietoris sequence for  $\mathrm{Tor}^Q$  in terms of  $\mathrm{Tor}^S$ ,  $\mathrm{Tor}^R$  and  $\mathrm{Tor}^T$ . If  $R$  is left noetherian and  $S$  and  $T$  are left coherent, then take a set  $(F_i)_{i \in I}$  of right flat  $Q$ -modules and compare the Mayer–Vietoris sequence for  $\mathrm{Tor}(\prod F_i, M)$  with the product of those for  $\mathrm{Tor}(F_i, M)$ , for an arbitrary left  $Q$ -module  $M$ . This gives  $\mathrm{Tor}_i^Q(\prod F_i, M) = 0$  for  $i > 1$ . Since  $S$  and  $T$  are left coherent and that, as  $R$  is left noetherian and the  $F_i$  are right flat  $R$ -modules,  $(\prod F_i) \otimes_R M \rightarrow \prod (F_i \otimes_R M)$  is injective by [Å82] Lemma 6. It follows that  $\mathrm{Tor}_1^Q(\prod F_i, M)$  also vanishes, so that  $\mathrm{Tor}_i^Q(\prod F_i, M) = 0$  for all  $i > 0$  as required.  $\square$

Combining with Theorem 3.7 we get:

COROLLARY 4.4. *Suppose that  $H$  is a  $p$ -adic analytic group that is an amalgamated product of two compact open subgroups. Then  $A\langle H \rangle$  is coherent.*  $\square$

THEOREM 4.5. *Suppose that  $H$  is a  $p$ -adic analytic group that is an amalgamated product of two compact open subgroups. Then the category of finitely presented  $K$ -finite  $A$ -representations of  $H$  is an abelian subcategory of the category of  $A$ -representations of  $H$ .*

*Proof.* It suffices to show that the kernel or cokernel of a map of finitely presented  $K$ -finite  $A$ -representations of  $H$  is also a finitely presented  $K$ -finite  $A$ -representation. This is straightforward for cokernels, and does not require the ring  $A\langle H \rangle$ . For kernels, suppose that  $f : V \rightarrow W$  is a map of finitely presented  $K$ -finite  $A$ -representations of  $H$ . Then  $\ker(f)$  is a  $K$ -finite  $A$ -representation

of  $H$ , and by Proposition 3.8 and Corollary 4.4 it is finitely presented as a left  $A\langle H\rangle$ -module. By Proposition 3.8 again, it is a finitely presented  $A$ -representation of  $H$ .  $\square$

## 5 APPLICATIONS

Let  $F$  be a local field of characteristic 0 with ring of integers  $\mathcal{O}_F$  and residue field  $k$  of characteristic  $p$ , and let  $D$  be a division algebra over  $F$  with ring of integers  $\mathcal{O}_D$ . Choose a uniformiser  $\pi$  of  $D$ . Let  $G = GL_2(D)$  and let  $G' = SL_2(D)$  be the subgroup of elements of reduced norm 1. Let  $K_1 = GL_2(\mathcal{O}_D)$  and let  $K'_1 = SL_2(\mathcal{O}_D) = K'_1 \cap SL_2(D)$ . Let  $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \in G$ , and let  $K_2 = \alpha K_1 \alpha^{-1}$  and  $K'_2 = K_2 \cap G'$ . Let

$$I = K_1 \cap K_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_1 : c \equiv 0 \pmod{\pi} \right\}$$

and  $I' = I \cap G' = K'_1 \cap K'_2$ .

**THEOREM 5.1.** *The category of finitely presented  $K$ -finite  $A$ -representations of  $G'$  is an abelian subcategory of  $\mathcal{C}_A^{K\text{-fin}}(G')$ .*

*Proof.* By a theorem of Ihara (Serre [Ser77] Chapter II Corollary 1) we know that  $G' = K'_1 *_I K'_2$ . The theorem follows from Theorem 4.5.  $\square$

**COROLLARY 5.2.** *The category of finitely presented,  $K$ -finite, (locally)  $Z$ -finite  $A$ -representations of  $G$  is an abelian subcategory of  $\mathcal{C}_A^{K\text{-fin}}(G)$ .*

*Proof.* Let  $G^0$  be the subgroup of  $G$  of elements whose reduced norm is in  $\mathcal{O}_F^\times$  and let  $Z$  be the centre of  $G$ . Then  $ZG^0$  has finite index in  $G$ ,  $Z \cap G^0$  is compact, and  $Z/Z \cap K$  is finitely generated for any compact open subgroup  $K$  of  $G$ . Let  $f : V_1 \rightarrow V_2$  be a map of  $K$ -finite  $Z$ -finite finitely presented representations of  $G$ . By Proposition 2.10 they are finitely presented representations of  $G^0$ . By [Ser77] Chapter II Theorem 3,  $G^0 = K_1 *_I K_2$ , and so Theorem 4.5 the kernel  $\ker(f)$  is finitely presented as a representation of  $G^0$ . By Proposition 2.10 again, it is a finitely presented representation of  $G$ .  $\square$

## REFERENCES

- [CEG<sup>+</sup>16] Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paškūnas, and Sug Woo Shin, *Patching and the  $p$ -adic local Langlands correspondence*, Camb. J. Math. 4 (2016), no. 2, 197–287. MR 3529394
- [Cha60] Stephen U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc. 97 (1960), 457–473. MR 0120260

- [Eme10] Matthew Emerton, *Ordinary parts of admissible representations of  $p$ -adic reductive groups I. Definition and first properties*, Astérisque (2010), no. 331, 355–402. MR 2667882
- [Hu12] Yongquan Hu, *Diagrammes canoniques et représentations modulo  $p$  de  $GL_2(F)$* , J. Inst. Math. Jussieu 11 (2012), no. 1, 67–118. MR 2862375
- [Koh17] Jan Kohlhaase, *Smooth duality in natural characteristic*, Adv. Math. 317 (2017), 1–49. MR 3682662
- [Å82] Hans Åberg, *Coherence of amalgamations*, J. Algebra 78 (1982), no. 2, 372–385. MR 680365
- [Sch15] Benjamin Schraen, *Sur la présentation des représentations supersingulières de  $GL_2(F)$* , J. Reine Angew. Math. 704 (2015), 187–208. MR 3365778
- [Ser77] Jean-Pierre Serre, *Arbres, amalgames,  $SL_2$* , Société Mathématique de France, Paris, 1977. Avec un sommaire anglais. Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46. MR 0476875
- [Sta17] The Stacks Project Authors, *Stacks Project*, <http://stacks.math.columbia.edu>, 2017.
- [Vig11] Marie-France Vigneras, *Le foncteur de Colmez pour  $GL(2, F)$* , Arithmetic geometry and automorphic forms, Adv. Lect. Math. (ALM), vol. 19, Int. Press, Somerville, MA, 2011, pp. 531–557. MR 2906918

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